A Simple Controller for a Variable Stiffness Joint with Uncertain Dynamics and Prescribed Performance Guarantees*

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Abstract— **In this paper a simple tracking controller for a variable stiffness joint is proposed. System dynamics is considered unknown. The controller guarantees link and stiffness motor position performance specifications that have been a-priori set, utilizing full state feedback. Simulation results on the previously published CompAct-VSA joint validate the efficiency of the proposed control approach.**

Index Terms— **Variable stiffness joint, prescribed performance control, uncertain system dynamics.**

I. INTRODUCTION

In recent years, robot joint compliance has been considered central into allowing the utilisation of robots in the vicinity or in collaboration with humans by reducing intrisically the risk of user injuries and robot damages [1], [2]. Variable joint compliance has been recently introduced by the development of variable stiffness actuators to be adopted in robot joints as opposed to the fixed compliant joint solutions [3]–[9]. Variable stiffness joints (VSJ) imply a more complex mechanical design as it incorporates two actuators in order to adjust simultaneously both the joint position and stiffness. Consequently, a more complicated dynamic model is involved particularly when a multi-dof kinematic structure has to be considered that in turn implies a challenging control task. The proposed control solutions have been thus far limited to complicated algorithms assuming system model knowledge or estimation aiming at decoupling the position and stiffness dynamics [10]–[15]. Furthermore, none of the previously proposed controllers can guarantee prescribed performance characteristics on the system's response, such as steady state error, rate of convergence and maximum overshoot.

A design methodology, called Prescribed Performance Control (PPC), that allows us to impose certain specifications on the performance of measurable signals of uncertain systems was introduced lately in [16]. PPC has been subsequently modified in [17] to achieve the required performance specifications without even using approximators to acquire information concerning the considered system dynamics. The latter methodology was recently utilized to design a controller for a flexible joint robot with variable joint stiffness that however neglects the mechanical system dynamics which realizes the stiffness variation [18].

In this paper we propose a full state feedback, approximation free controller that guarantees prescribed performance attributes on the link and stiffness motor position tracking error of a variable stiffness joint. The proposed control design, although it is limited to a single joint case, is to our knowledge the only work that guarantees prescribed performance. Simulation results on the CompAct-VSA [5] illustrate the efficiency of the proposed control scheme.

II. PRESCRIBED PERFORMANCE PRELIMINARIES

For completeness and compactness of presentation this subsection summarizes preliminary knowledge on prescribed performance originally stated in [16]. In that respect, consider a generic tracking error $e(t) = \begin{bmatrix} e_1(t) & \dots & e_m(t) \end{bmatrix}^\intercal \in \mathbb{R}^m$. Prescribed performance is achieved if each element $e_i(t),\ i = 1, \dots, m$ evolves strictly within a predefined region that is bounded by a decaying function of time. The mathematical expression of prescribed performance is given, $\forall t \geq 0$, by the following inequalities:

$$\left.\begin{array}{ll} -M_i\rho_i(t) < e_i(t) < \rho_i(t), & e_i(0) \geq 0 \\ -\rho_i(t) < e_i(t) < M_i\rho_i(t), & e_i(0) \leq 0 \end{array}\right\} i = 1, \dots, m \quad (1)$$

where $0 \leq M_i \leq 1,\ i = 1, \dots, m$ and $\rho_i(t),\ i = 1, \dots, m$ are bounded, smooth, strictly positive and decreasing functions, implying a bounded first derivative, satisfying $\lim_{t\to\infty} \rho_i(t) = \rho_{i\infty} > 0,\ i = 1, \dots, m$, called performance functions [16]. As (1) implies, only one set of the performance bounds is employed and specifically the one associated with the sign of $e_i(0)$. The aforementioned statements are clearly illustrated in Fig. 1, for an exponential performance function

$$\rho_i(t) = (\rho_{i0} - \rho_{i\infty}) \exp(-l_i t) + \rho_{i\infty},\ i = 1, \dots, m, \quad (2)$$

with ρ_{i0}, $\rho_{i\infty}$, $l_i,\ i = 1, \dots, m$ strictly positive constants. The constant $\rho_{i0} = \rho_i(0),\ i = 1, \dots, m$ is selected such that (1) is satisfied at $t = 0$ (i.e., $\rho_i(0) > e_i(0)$ in case $e_i(0) \geq 0$ or $\rho_i(0) > -e_i(0)$ in case $e_i(0) \leq 0$). The constant $\rho_{i\infty} = \lim_{t\to\infty} \rho_i(t),\ i = 1, \dots, m$ represents the maximum allowable size of $e_i(t)$ at the steady state that can be set arbitrarily small to a value reflecting the resolution of the measurement device, thus achieving practical convergence of $e_i(t)$ to zero. Furthermore, the decreasing rate of $\rho_i(t),\ i = 1, \dots, m$, which is related to the constant $l_i,\ i = 1, \dots, m$ in this case, introduces a lower bound on the required speed of convergence of $e_i(t)$. Moreover, the

*This research is co-financed by the EU-ESF and Greek national funds through the Operational Program "Education and Lifelong Learning" of the National Strategic Reference Framework (NSRF)-Research Funding Program ARISTEIA I

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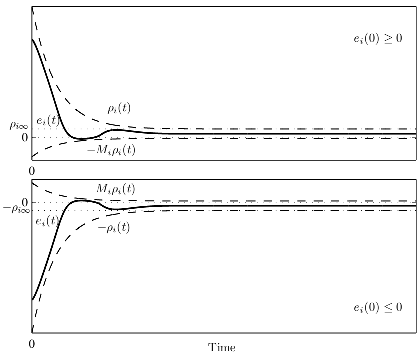

Fig. 1: Prescribed performance.

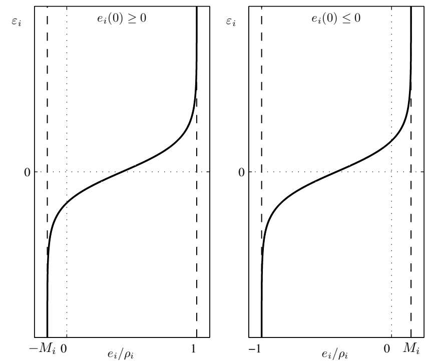

Fig. 2: The transformation function.

maximum overshoot is prescribed less than $M_i\rho_i(0)$, $i = 1, \ldots, m$, which may even become zero by setting $M_i = 0$, $i = 1, \ldots, m$. Thus, the appropriate selection of the performance function $\rho_i(t)$, $i = 1, \ldots, m$, as well as of the constant M_i, $i = 1, \ldots, m$, imposes performance bounds for the tracking error $e_i(t)$, $i = 1, \ldots, m$.

To introduce prescribed performance, an error transformation is incorporated modulating the tracking error element $e_i(t)$, $i = 1, \ldots, m$ with respect to the required performance bounds imposed by $\rho_i(t)$, M_i, $i = 1, \ldots, m$. More specifically, we define:

$$\varepsilon_i(t) = T_i\left(\frac{e_i(t)}{\rho_i(t)}\right), \ i = 1, \ldots, m \tag{3}$$

where $\varepsilon_i(t)$, $i = 1, \ldots, m$ is the transformed error and $T_i(\cdot)$, $i = 1, \ldots, m$ is a smooth, strictly increasing function defining a bijective mapping:

$$\left.\begin{array}{ll} T_i : (-M_i, 1) \to (-\infty, \infty), & e_i(0) \geq 0 \\ T_i : (-1, M_i) \to (-\infty, \infty), & e_i(0) \leq 0 \end{array}\right\} \ i = 1, \ldots, m. \tag{4}$$

A candidate transformation function, illustrated in Fig 2, could be

$$T_i\left(\frac{e_i(t)}{\rho_i(t)}\right) = \begin{cases} a_i \ln\left(\dfrac{M_i + \frac{e_i(t)}{\rho_i(t)}}{1 - \frac{e_i(t)}{\rho_i(t)}}\right), \text{in case } e_i(0) \geq 0 \\ a_i \ln\left(\dfrac{1 + \frac{e_i(t)}{\rho_i(t)}}{M_i - \frac{e_i(t)}{\rho_i(t)}}\right), \text{in case } e_i(0) \leq 0 \end{cases} \tag{5}$$

$i = 1, \ldots, m$, where a_i are positive design constants. As (4) implies and the aforementioned example clarifies, the choice of the mapping $T_i(\cdot)$, $i = 1, \ldots, m$, depends only on the sign of $e_i(0)$, $i = 1, \ldots, m$. Notice also that since $\rho_i(0)$, $i = 1, \ldots, m$ is selected such that (1) is satisfied at $t = 0$, $\varepsilon_i(0)$, $i = 1, \ldots, m$ is finite owing to (4). The case of $e_i(0) = 0$ requires the choice of $M_i \neq 0$, $i = 1, \ldots, m$, since otherwise (i.e., $M_i = 0$) $\varepsilon_i(0)$, $i = 1, \ldots, m$ becomes infinite.

Owing to the properties of the error transformation, we satisfy the prescribed performance (1) for all $t \geq 0$, by keeping $\varepsilon_i(t)$, $i = 1, \ldots, m$ bounded. Notice that the bounds of $\varepsilon_i(t)$, $i = 1, \ldots, m$ do not affect the evolution of $e_i(t)$, $i = 1, \ldots, m$, which are solely prescribed by (1) and thus by the selection of the performance functions $\rho_i(t)$, $i = 1, \ldots, m$ as well as the constants M_i, $i = 1, \ldots, m$.

III. Problem Description and Controller Design

Let us consider a variable stiffness joint (VSJ) consisting of two modules, the stiffness actuator module which regulates the stiffness and the position actuator module which regulates the joint's link position. A reduced model of the aforementioned system can be expressed as follows [5]:

$$M\ddot{q} + C\dot{q} + g_l \sin q = \tau_E(\theta, q, \theta_k) \tag{6}$$

$$J_\theta\ddot{\theta} + B_\theta\dot{\theta} + \tau_E(\theta, q, \theta_k) = \tau_m \tag{7}$$

$$J_k\ddot{\theta}_k + B_k\dot{\theta}_k + \tau_R(\theta, q, \theta_k) = \tau_k \tag{8}$$

where $q \in \mathbb{R}$ is the link angle, $\theta \in \mathbb{R}$ is the link motor angular position, $\theta_k \in \mathbb{R}$ is the stiffness motor angular position, while $\dot{q}, \dot{\theta}, \dot{\theta}_k \in \mathbb{R}$ are the respective velocities. Moreover, $M \in \mathbb{R}$ is the link inertia, $C \in \mathbb{R}$ is the link friction coefficient, $J_\theta, J_k, B_\theta, B_k \in \mathbb{R}$ are the two motor inertias and damping coefficients respectively, the latter including both the physical damping and back emf damping of the motors. In addition, $g_l = mgl_c$, where m is the link mass, g is the gravity acceleration and l_c is the distance to the link center of mass. Moreover, τ_E is the elastic torque given by

$$\tau_E = K_E\frac{\theta_k^2}{(\Delta - n\theta_k)^2}(\theta - q) \tag{9}$$

where K_E, Δ, n are positive constants and τ_R is the resistant torque given by

$$\tau_R = K_R\frac{\theta_k(\theta - q)^2}{(\Delta - n\theta_k)^3} \tag{10}$$

where K_R is a positive constant. A schematic of the actuator model and operation principle is shown in Fig. 3. For the

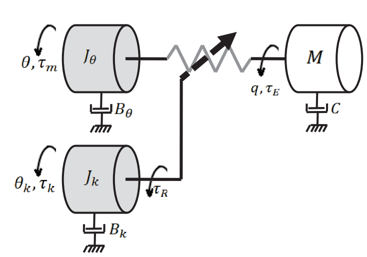

Fig. 3: Schematic of the actuator model and operation principle.

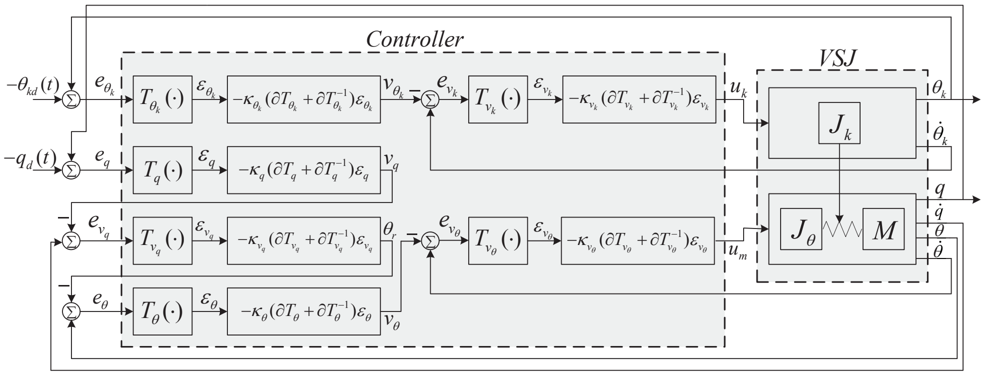

Fig. 4: The proposed control scheme.

joint stiffness $K_E \frac{\theta_k^2}{(\Delta - n\theta_k)^2}$ in (9), it is considered

$$K_E \frac{\theta_k^2(t)}{(\Delta - n\theta_k(t))^2} \geq k > 0, \forall t \geq 0 \tag{11}$$

where k a possibly small constant and

$$\Delta > n\theta_k(t), \forall t \geq 0 \tag{12}$$

Both (11), (12), which are imposed by construction and low level firmware limits [5], are required to establish a well-defined joint stiffness representation as well as a controllable VSJ model (6)-(8). The latter is clarified by noticing that a direct consequence of (11) is $|\theta_k(t)| > 0, \forall t \geq 0$; thus avoiding dropping stiffness to zero, which would lead to an unactuated system. Finally $\tau_m, \tau_k \in \mathbb{R}$ are the motor torques after the reduction drives which are given by $\tau_m = k_\theta u_m$ $\tau_k = k_k u_k$ where u_m, u_k are the input voltages and k_θ, k_k are positive electrical motor constants.

For the VSJ system (6)-(12), the state $x = (q, \dot{q}, \theta, \dot{\theta}, \theta_k, \dot{\theta}_k)$ is assumed to be available for measurement, while its dynamics is assumed to be unknown.

Our goal is to design a state feedback controller to control the outputs q, θ_k of the VSJ to track a given, smooth and bounded reference trajectory $q_d(t) \in \mathbb{R}$, $\theta_{kd}(t) \in \mathbb{R}$ respectively, with prescribed performance. This means that the output error should be guaranteed to converge to a predefined arbitrarily small residual set, with rate less than a prespecified value, exhibiting maximum overshoot less than a sufficiently small preassigned constant. Moreover, all other closed loop signals should be kept bounded. The above task shall be referred to as the *Prescribed Performance Control for a Variable Stiffness Joint (PPC/VSJ) problem.*

Remark 1: The stiffness actuator desired trajectory $\theta_{kd}(t)$ corresponds to a desired joint stiffness trajectory that is assumed to be provided by a suitable safety or interaction control level on the basis of the requirements set by the particular application and context. For example, a low velocity-high stiffness policy was utilized in [7], [19]. The issue of exactly determining $\theta_{kd}(t)$ goes beyond the scope of this paper to be addressed in detail.

A. The Proposed Control Scheme

Let us define $\varepsilon_i,\ i \in \{q, \theta_k, v_q, \theta, v_\theta, v_k\}$ as

$$\varepsilon_i = T_i\left(\frac{e_i}{\rho_i}\right) \in \mathbb{R}, \tag{13}$$

with $e_q = q - q_d \in \mathbb{R}, e_{\theta_k} = \theta_k - \theta_{kd} \in \mathbb{R}, e_{v_q} = \dot{q} - v_q \in \mathbb{R}, e_\theta = \theta - \theta_r \in \mathbb{R}, e_{v_\theta} = \dot{\theta} - v_\theta \in \mathbb{R}, e_{v_k} = \dot{\theta}_k - v_{\theta_k} \in \mathbb{R}$ where $v_q, \theta_r, v_\theta, v_{\theta_k}$ are intermediate control signals defined below and

$$\vartheta T_i = \frac{1}{\rho_i} \frac{\partial T_i}{\partial\left(\frac{e_i}{\rho_i}\right)} \in \mathbb{R}, i \in \{q, \theta_k, v_q, \theta, v_\theta, v_k\} \tag{14}$$

Notice that $\vartheta T_i > 0$. A solution to the considered problem, is provided by

$$\begin{aligned}
u_m(t) &= -\kappa_{v_\theta}\left(\vartheta T_{v_\theta} + \vartheta T_{v_\theta}^{-1}\right)\varepsilon_{v_\theta} && (15)\\
v_\theta(t) &= -\kappa_\theta\left(\vartheta T_\theta + \vartheta T_\theta^{-1}\right)\varepsilon_\theta && (16)\\
\theta_r(t) &= -\kappa_{v_q}\left(\vartheta T_{v_q} + \vartheta T_{v_q}^{-1}\right)\varepsilon_{v_q} && (17)\\
v_q(t) &= -\kappa_q\left(\vartheta T_q + \vartheta T_q^{-1}\right)\varepsilon_q && (18)\\
u_k(t) &= -\kappa_{v_k}\left(\vartheta T_{v_k} + \vartheta T_{v_k}^{-1}\right)\varepsilon_{v_k} && (19)\\
v_{\theta_k}(t) &= -\kappa_{\theta_k}\left(\vartheta T_{\theta_k} + \vartheta T_{\theta_k}^{-1}\right)\varepsilon_{\theta_k} && (20)
\end{aligned}$$

where, in (15)-(20), $\kappa_{v_\theta}, \kappa_\theta, \kappa_{v_q}, \kappa_q, \kappa_{v_k}$ and κ_{θ_k} are positive control gains. Fig. 4 illustrates the proposed control scheme. The following theorem summarizes the main results of the paper.

Theorem 1: Consider a variable stiffness joint (6)-(12). The controller (15)-(20) with the signals $\varepsilon_i, \vartheta T_i,\ i \in \{q, \theta_k, v_q, \theta, v_\theta, v_k\}$ as defined in (13), (14), solves the PPC/VSJ problem.

Proof. The proof of Theorem 1 can be found in the Appendix. □

The proposed controller achieves any performance requirement for the system output (q, θ_k) regarding the steady state error, the speed of convergence as well as the overshoot using state feedback and without requesting any knowledge of the system nonlinearities. Experiments verifying the prescribed performance control methodology have already been conducted on a single rigid link robot [20] and from the experience gained thus far the most significant implementation

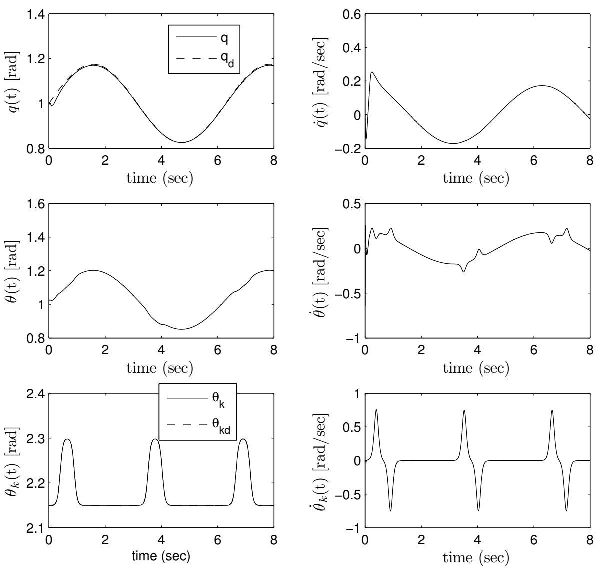

Fig. 5: Link and motors angle evolution.

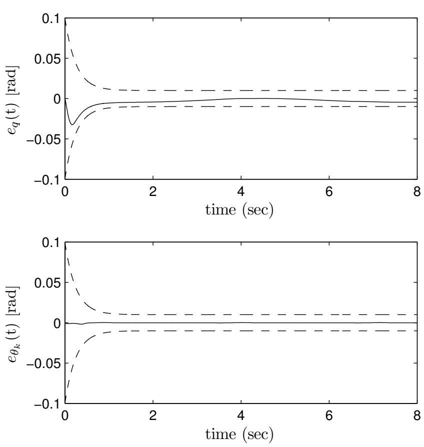

Fig. 6: Output errors (solid lines) along with performance bounds (dashed lines).

issue that arises in this type of controllers is related to their discretization which can however be resolved if sufficiently fast sampling frequency is allowed. To the best of the authors knowledge, the controller summarized in Theorem 1 is by far the simplest architecture reported in the relevant literature capable of succeeding such a demanding task for a variable stiffness joint.

IV. Simulation Results

In order to demonstrate the proposed control scheme we consider the compact variable stiffness actuator presented in [5], the CompAct-VSA. Link friction is neglected ($C = 0$) while the values of the remaining model parameters are given in Table I.

We set $x(0) = [1\ 0\ 1.0248\ 0\ 2.15\ 0]^{\mathsf{T}}$. The initial value of θ_k corresponds to a stiffness value of $200\mathrm{Nm/rad}$. Our purpose is to force the outputs q, θ_k to track the smooth, bounded trajectories $q_d(t) = 1 + 0.1745 \sin t$ and $\theta_{kd}(t) = 2.3 + \sum_{i=1}^{7} (-1)^i 0.15/\left[\left(1 + e^{-20(t-\sigma_i)}\right)\right]$ where $\sigma_1 = -0.3, \sigma_2 = 0.4, \sigma_3 = 0.906, \sigma_4 = 3.526, \sigma_5 = 4.032, \sigma_6 = 6.652, \sigma_7 = 7.158$ corresponding to a desired stiffness of the same form between the values of 170 and $582 Nm/rad$. For the output errors $e_q(t)$ and $e_{\theta_k}(t)$ we require a steady state of no more than $0.01\mathrm{rad}$ and minimum speed of convergence as obtained by the exponential e^{-4t}. As the theoretical analysis dictates, the performance requirements for the rest of the states don't need to be as strict. The aforementioned transient and steady state error bounds are prescribed via performance functions defined as in (2), with parameters as seen in Table II. Furthermore, the output and state error

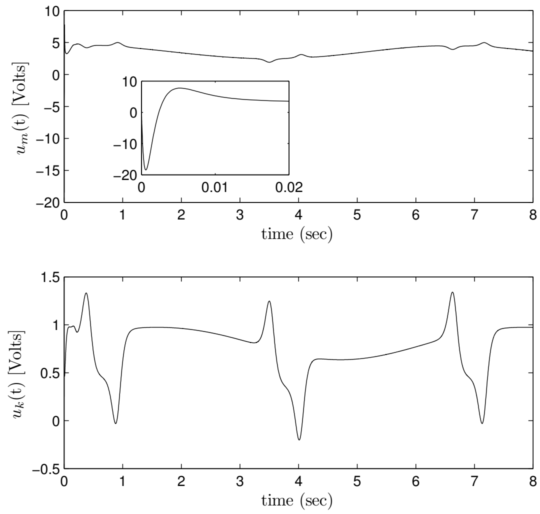

Fig. 7: Control inputs.

TABLE I: Model system parameters

Parameter	Value	Unit
M	0.162	kgm^2
J_θ	0.0575	kgm^2
J_k	0.0062	kgm^2
B_θ	7.7781	Nms/rad
B_k	0.5262	Nms/rad
m	2	kg
l_c	0.3	m
K_E	$1.62 \cdot 10^{-4}$	Nm^2/rad^2
K_R	$2.4 \cdot 10^{-6}$	Nm^3/rad^2
Δ	0.015	m
n	0.006	-
k_m	1.4139	$Nm/A\Omega$
k_k	0.6014	$Nm/A\Omega$

TABLE II: Prescribed performance function parameters

	ρ_0	ρ_∞	l	M
q	0.1	0.01	4	1
$\dot{q}$	10	6	2	1
θ	2	1.7	16.26	0
$\dot{\theta}$	9	7	13.4	1
θ_κ	0.1	0.01	4	1
$\dot{\theta}_\kappa$	14	12	5.92	1

transformations are chosen as in (5). We simulate the closed loop system using the design constants $\kappa_q = 19.85, \kappa_{v_q} = 0.2, \kappa_\theta = 1.1, \kappa_{v_\theta} = 16.88, \kappa_{\theta_k} = 16.72, \kappa_{v_k} = 8.66, \alpha_q = 0.018, \alpha_{v_q} = 0.6, \alpha_\theta = 13.5, \alpha_{v_\theta} = 12, \alpha_{\theta_k} = 0.078, \alpha_{v_k} = 3$. As the output error performance is solely determined by performance specifications, the selection of the above design constants is made by adopting those values that lead to reasonable control effort so that the maximum motor power supply is not exceeded. The simulation results are depicted in Figures 5-7. Figure 5 shows the link and motors angle evolution. The output errors clearly satisfy the prescribed performance specifications as illustrated in Figure 6. Finally the demanded control effort (input voltages) are pictured in Figure 7. It is clear that the control effort is reasonable for such a control task and well beneath the maximum motor power supply voltage which for this case is $24V$ for both motors [5].

V. Conclusions

In this work a state feedback controller is proposed for a single variable stiffness joint achieving tracking of link and motor stiffness angles with any pre-set performance requirements, without requesting any system knowledge. Simulation results reveal that such a demanding task can be achieved with reasonable control effort. Future work includes experiments with the system [5] as well as consideration of multi variable stiffness joint robots.

Appendix

We initially formulate the closed loop system dynamics in the transformed error space. Employing the definitions of the output and state errors $e_q, e_{\theta_k}, e_{v_q}, e_\theta, e_{v_\theta}, e_{v_k}$, the inverse transformations

$$e_i = \rho_i T_i^{-1}(\varepsilon_i), \; i \in \{q, \theta_k, v_q, \theta, v_\theta, v_k\} \tag{21}$$

as well as (16)-(18),(20) it is straightforwardly obtained:

$$\begin{aligned} q &= q_d(t) + e_q && (22)\\ &= q_d(t) + \rho_q T_q^{-1}(\varepsilon_q) && (23)\\ \dot{q} &= v_q(t) + e_{v_q} && (24)\\ &= -\kappa_q(\vartheta T_q + \vartheta T_q^{-1})\varepsilon_q + \rho_{v_q} T_{v_q}^{-1}(\varepsilon_{v_q}) && (25)\\ \theta &= \theta_r(t) + e_\theta && (26)\\ &= -\kappa_{v_q}(\vartheta T_{v_q} + \vartheta T_{v_q}^{-1})\varepsilon_{v_q} + \rho_\theta T_\theta^{-1}(\varepsilon_\theta) && (27)\\ \dot{\theta} &= v_\theta(t) + e_{v_\theta} && (28)\\ &= -\kappa_\theta(\vartheta T_\theta + \vartheta T_\theta^{-1})\varepsilon_\theta + \rho_{v_\theta} T_{v_\theta}^{-1}(\varepsilon_{v_\theta}) && (29)\\ \theta_k &= \theta_{kd}(t) + e_{\theta_k} && (30)\\ &= \theta_{kd}(t) + \rho_{\theta_k} T_{\theta_k}^{-1}(\varepsilon_{\theta_k}) && (31)\\ \dot{\theta}_k &= v_{\theta_k}(t) + e_{v_k} && (32)\\ &= -\kappa_{\theta_k}(\vartheta T_{\theta_k} + \vartheta T_{\theta_k}^{-1})\varepsilon_{\theta_k} + \rho_{v_k} T_{v_k}^{-1}(\varepsilon_{v_k}). && (33) \end{aligned}$$

The link and motor accelerations with respect to the transformed errors are derived as

$$\begin{aligned} \ddot{q} &= -M^{-1}\mathcal{Z}_q(\varepsilon_q, \varepsilon_{v_q}, t) \\ &\quad -M^{-1}\mathcal{Z}_{E_q}(\varepsilon_{\theta_k}, t)(q - \theta) && (34)\\ \ddot{\theta} &= -J_\theta^{-1}\mathcal{Z}_\theta(\varepsilon_\theta, \varepsilon_{v_\theta}, t) - J_\theta^{-1}\mathcal{Z}_{E_\theta}(\varepsilon_{\theta_k}, \varepsilon_\theta, \varepsilon_q, \varepsilon_{v_q}, t) \\ &\quad + J_\theta^{-1} k_\theta u_m && (35)\\ \ddot{\theta}_k &= -J_k^{-1}\mathcal{Z}_{\theta_k}(\varepsilon_{\theta_k}, \varepsilon_{v_k}, t) - J_k^{-1}\mathcal{Z}_k(\varepsilon_{\theta_k}, \varepsilon_\theta, \varepsilon_q, \varepsilon_{v_q}, t) \\ &\quad + J_k^{-1} k_k u_k && (36) \end{aligned}$$

where

$$\begin{aligned} \mathcal{Z}_q(\varepsilon_q, \varepsilon_{v_q}, t) &= \mathcal{F}_q(q(\varepsilon_q, t), \dot{q}(\varepsilon_q, \varepsilon_{v_q}, t)) && (37)\\ \mathcal{Z}_{E_q}(\varepsilon_{\theta_k}, t) &= \mathcal{F}_{E_q}(\theta_k(\varepsilon_{\theta_k}, t)) && (38)\\ \mathcal{Z}_\theta(\varepsilon_\theta, \varepsilon_{v_\theta}, t) &= \mathcal{F}_\theta(\dot{\theta}(\varepsilon_\theta, \varepsilon_{v_\theta}, t)) && (39)\\ \mathcal{Z}_{E_\theta}(\varepsilon_{\theta_k}, \varepsilon_\theta, \varepsilon_q, \varepsilon_{v_q}, t) &= \tau_E(q(\varepsilon_q, t), \theta(\varepsilon_{v_q}, \varepsilon_\theta, t), \\ &\quad \theta_k(\varepsilon_{\theta_k}, t)) && (40)\\ \mathcal{Z}_{\theta_k}(\varepsilon_{\theta_k}, \varepsilon_{v_k}, t) &= \mathcal{F}_{\theta_k}(\dot{\theta}_k(\varepsilon_{\theta_k}, \varepsilon_{v_k}, t)) && (41) \end{aligned}$$

$$\begin{aligned} \mathcal{Z}_k(\varepsilon_{\theta_k}, \varepsilon_\theta, \varepsilon_q, \varepsilon_{v_q}, t) &= \tau_R(q(\varepsilon_q, t), \theta(\varepsilon_{v_q}, \varepsilon_\theta, t), \\ &\quad \theta_k(\varepsilon_{\theta_k}, t)) && (42) \end{aligned}$$

with $\mathcal{F}_q(q, \dot{q}) = C\dot{q} + g_l \sin q$, $\mathcal{F}_{E_q}(\theta_k) = K_E \frac{\theta_k^2}{(\Delta - n\theta_k)^2}$, $\mathcal{F}_\theta(\dot{\theta}) = B_\theta \dot{\theta}$ and $\mathcal{F}_{\theta_k}(\dot{\theta}_k) = B_k \dot{\theta}_k$.

Differentiating (13) with respect to time and using (14), we obtain:

$$\dot{\varepsilon}_i = \frac{\partial T_i}{\partial\left(\frac{e_i}{\rho_i}\right)} \frac{d}{dt}\left(\frac{e_i}{\rho_i}\right) = \vartheta T_i \; (\dot{e}_i - \nu_i(\varepsilon_i, t)) \tag{43}$$

where $\nu_i(\varepsilon_i, t)$ is defined using (21) as follows:

$$\nu_i(\varepsilon_i, t) = e_i \frac{\dot{\rho}_i}{\rho_i} = \dot{\rho}_i(t) T_i^{-1}(\varepsilon_i), \tag{44}$$

with $i \in \{q, \theta_k, v_q, \theta, v_\theta, v_k\}$. Substituting (21), (22), (24), (26), (28), (30), (32) in (43) using the error definitions yields:

$$\begin{aligned} \dot{\varepsilon}_q &= \vartheta T_q \; (\dot{e}_q - \nu_q(\varepsilon_q, t)) \\ &= \vartheta T_q \; (\dot{q} - \dot{q}_d(t) - \nu_q(\varepsilon_q, t)) \\ &= \vartheta T_q \; (v_q(t) + \rho_{v_q} T_{v_q}^{-1}(\varepsilon_{v_q}) - \dot{q}_d(t) - \nu_q(\varepsilon_q, t)) \\ \dot{\varepsilon}_{\theta_k} &= \vartheta T_{\theta_k} \; (\dot{e}_{\theta_k} - \nu_{\theta_k}(\varepsilon_{\theta_k}, t)) \\ &= \vartheta T_{\theta_k} \; (\dot{\theta}_k - \dot{\theta}_{kd}(t) - \nu_{\theta_k}(\varepsilon_{\theta_k}, t)) \\ &= \vartheta T_{\theta_k} \; (v_{\theta_k}(t) + \rho_{v_k} T_{v_k}^{-1}(\varepsilon_{v_k}) - \dot{\theta}_{kd}(t) - \nu_{\theta_k}(\varepsilon_{\theta_k}, t)) \\ \dot{\varepsilon}_{v_q} &= \vartheta T_{v_q} \; (\dot{e}_{v_q} - \nu_{v_q}(\varepsilon_{v_q}, t)) \\ &= \vartheta T_{v_q} \; (\ddot{q} - \dot{v}_q(t) - \nu_{v_q}(\varepsilon_{v_q}, t)) \\ \dot{\varepsilon}_\theta &= \vartheta T_\theta \; (\dot{e}_\theta - \nu_\theta(\varepsilon_\theta, t)) \\ &= \vartheta T_\theta \; (\dot{\theta} - \dot{\theta}_r(t) - \nu_\theta(\varepsilon_\theta, t)) \\ &= \vartheta T_\theta \; (v_\theta(t) + \rho_{v_\theta} T_{v_\theta}^{-1}(\varepsilon_{v_\theta}) - \dot{\theta}_r(t) - \nu_\theta(\varepsilon_{v_\theta}, t)) \\ \dot{\varepsilon}_{v_\theta} &= \vartheta T_{v_\theta} \; (\dot{e}_{v_\theta} - \nu_{v_\theta}(\varepsilon_{v_\theta}, t)) \\ &= \vartheta T_{v_\theta} \; (\ddot{\theta} - \dot{v}_\theta(t) - \nu_{v_\theta}(\varepsilon_{v_\theta}, t)) \\ \dot{\varepsilon}_{v_k} &= \vartheta T_{v_k} \; (\dot{e}_{v_k} - \nu_{v_k}(\varepsilon_{v_k}, t)) \\ &= \vartheta T_{v_k} \; (\ddot{\theta}_k - \dot{v}_{\theta_k}(t) - \nu_{v_k}(\varepsilon_{v_k}, t)) \end{aligned}$$

Finally, substituting the controller (15) - (20) and the system dynamics (34) - (36) the closed loop dynamics in the transformed error space can be written as follows:

$$\dot{\varepsilon}_q = -\kappa_q \varepsilon_q - \kappa_q \vartheta T_q^2 \varepsilon_q - \vartheta T_q \left(\dot{q}_d + \nu_q(\varepsilon_q, t) \right) + \vartheta T_q \rho_{v_q} T_{v_q}^{-1}(\varepsilon_{v_q}) \quad (45)$$

$$\dot{\varepsilon}_{\theta_k} = -\kappa_{\theta_k} \varepsilon_{\theta_k} - \kappa_{\theta_k} \vartheta T_{\theta_k}^2 \varepsilon_{\theta_k} - \vartheta T_{\theta_k} \left(\dot{\theta}_{kd} + \nu_{\theta_k}(\varepsilon_{\theta_k}, t) \right) + \vartheta T_{\theta_k} \rho_{v_k} T_{v_k}^{-1}(\varepsilon_{v_k}) \quad (46)$$

$$\begin{aligned} \dot{\varepsilon}_{v_q} = & -\kappa_{v_q} M^{-1} \mathcal{Z}_{E_q}(\varepsilon_{\theta_k}, t) \left\{ \varepsilon_{v_q} + \vartheta T_{v_q}^2 \varepsilon_{v_q} \right\} \\ & + \vartheta T_{v_q} \left\{ -M^{-1} \mathcal{Z}_q(\varepsilon_q, \varepsilon_{v_q}, t) - \dot{v}_q - \nu_{v_q}(\varepsilon_{v_q}, t) + \right. \\ & + M^{-1} \mathcal{Z}_{E_q}(\varepsilon_{\theta_k}, t)(\rho_\theta T_\theta^{-1}(\varepsilon_\theta) \\ & \left. - q_d - \rho_q T_q^{-1}(\varepsilon_q)) \right\} \quad (47) \end{aligned}$$

$$\dot{\varepsilon}_\theta = -\kappa_\theta \varepsilon_\theta - \kappa_\theta \vartheta T_\theta^2 \varepsilon_\theta - \vartheta T_\theta \left(\dot{\theta}_r + \nu_\theta(\varepsilon_\theta, t) \right) + \vartheta T_\theta \rho_{v_\theta} T_{v_\theta}^{-1}(\varepsilon_{v_\theta}) \quad (48)$$

$$\begin{aligned} \dot{\varepsilon}_{v_\theta} = & -J_\theta^{-1} k_\theta \kappa_{v_\theta} (\varepsilon_{v_\theta} + \vartheta T_{v_\theta}^2 \varepsilon_{v_\theta}) + \\ & + \vartheta T_{v_\theta} \left\{ -J_\theta^{-1} \mathcal{Z}_\theta(\varepsilon_\theta, \varepsilon_{v_\theta}, t) \right. \\ & \left. - J_\theta^{-1} \mathcal{Z}_{E_\theta}(\varepsilon_{\theta_k}, \varepsilon_\theta, \varepsilon_q, \varepsilon_{v_q}, t) - \dot{v}_\theta - \nu_{v_\theta}(\varepsilon_{v_\theta}, t) \right\} \quad (49) \end{aligned}$$

$$\begin{aligned} \dot{\varepsilon}_{v_k} = & -J_k^{-1} k_k \kappa_{v_k} (\varepsilon_{v_k} + \vartheta T_{v_k}^2 \varepsilon_{v_k}) + \\ & + \vartheta T_{v_k} \left\{ -J_k^{-1} \mathcal{Z}_{\theta_k}(\varepsilon_{\theta_k}, \varepsilon_{v_\theta k}, t) \right. \\ & \left. - J_k^{-1} \mathcal{Z}_k(\varepsilon_{\theta_k}, \varepsilon_\theta, \varepsilon_q, \varepsilon_{v_q}, t) - \dot{v}_{\theta_k} - \nu_{v_k}(\varepsilon_{v_k}, t) \right\} \quad (50) \end{aligned}$$

According to the prescribed performance preliminaries, the *PPC/VSJ problem* is solved if the uniform boundedness of $\varepsilon_i,\ i = \{q, \theta_k, v_q, \theta, v_\theta, v_k\}$, is proved. The system (45)-(50) falls within the class of MIMO systems in block triangular form considered in [17]. Hence, the proof of Theorem 1 follows the steps of [17].

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